Journal of Engineering Physics and Thermophysics, Vol. 81, No. 6, 2008

ASYMPTOTIC ANALYSIS OF THE SOLUTION OF THE PROBLEM OF NONSTATIONARY HEAT CONDUCTION OF LAMINAR ANISOTROPIC INHOMOGENEOUS PLATES FOR SMALL BIOT NUMBERS ON FACES

Yu. V. Nemirovskii and A. P. Yankovskii UDC 536.21

The external asymptotic expansion of the solution of the problem of nonstationary heat conduction of laminar anisotropic inhomogeneous plates for small Biot numbers on faces has been constructed. The resulting two-dimensional resolving equations have been analyzed; the asymptotic properties of solutions of the heat-conduction problem have been investigated.

Introduction. The present work is a continuation of the investigations of [1–3] where the asymptotic expansions of the solution of a stationary problem of heat conduction of thin single-layer and laminar plates in a small parameter which is the ratio of the thickness of the structure to its characteristic dimension in plan were constructed. This investigation seeks to construct the asymptotic expansion of the solution of the problem of nonstationary heat conduction of laminar anisotropic plates with boundary conditions of the second and third kind on faces under the assumption that the Biot numbers on these faces are small compared to unity. Such a solution, in particular, will make it possible to evaluate the degree of accuracy with which a constant temperature or that distributed by the linear, square, and other laws over the thickness and layers of the plate can be prescribed in a thin-walled laminar structure beyond the boundary layer.

Formulation of the Problem of Heat Conduction of Laminar Plates. Let us consider a plate of constant thickness, which consists of M anisotropic inhomogeneous layers of a constant thickness, too. We tie a rectangular Cartesian coordinate system $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$ to the plate so that the reference plane $\overline{x}_3 = 0$ is coincident with the lower face plane of the plate. We number all layers successively from bottom to top, i.e., the first layer will be the lower layer, whereas the Mth layer will be the upper one. The conditions of ideal thermal contact are observed at the boundaries between the layers.

Let us introduce into consideration dimensionless variables, functions, and quantities:

$$x_i = \overline{x}_i / a \quad (i = 1, 2) , \quad x_3 = \overline{x}_3 / \overline{H} \quad (0 \le x_3 \le 1) , \quad t = \overline{t} / \overline{t}_* \quad (\overline{t}_* > 0) ,$$

$$T^{(m)} = \overline{T}^{(m)} / \overline{T}_* , \quad \lambda_{ij}^{(m)} = \overline{\lambda}_{ij}^{(m)} / \overline{\lambda}_* \quad (i, j = 1, 2, 3) , \quad Q^{(m)} = \overline{Q}^{(m)} a^2 / (\overline{\lambda}_* \overline{T}_*) ,$$

$$T_\infty^{(\pm)} = \overline{T}_\infty^{(\pm)} / \overline{T}_* , \quad Q^{(\pm)} = \overline{Q}^{(\pm)} a / (\overline{\lambda}_* \overline{T}_*) , \quad \alpha_{(\pm)} = \overline{\alpha}_{(\pm)} a / \overline{\lambda}_* , \quad T_\infty = \overline{T}_\infty / \overline{T}_* , \quad T_0 = \overline{T}_0 / \overline{T}_* , \tag{1}$$

$$q_n^{(m)} = \overline{q}_n^{(m)} a / (\overline{\lambda}_* \overline{T}_*) , \quad \alpha^{(m)} = \overline{\alpha}^{(m)} a / \overline{\lambda}_* , \quad C^{(m)} = \overline{c}^{(m)} \overline{\rho}^{(m)} a^2 / (\overline{\lambda}_* \overline{t}_*) , \quad t_0 = \overline{t}_0 / \overline{t}_* ,$$

$$H_m = \overline{H}_m / \overline{H} \quad (1 \le m \le M) , \quad H_0 = 0 , \quad H_M = 1 , \quad \varepsilon = \overline{H} / a .$$

The nonstationary problem of heat conduction of the laminar plate with account for (1) is described by the following dimensionless equations and relations: the equation of heat conduction of the mth layer

S. A. Khristianovich Institute of Theoretical and Applied Mechanics, Siberian Branch of the Russian Academy of Sciences, 4/1 Institutskaya Str., Novosibirsk, 630090, Russia. Translated from Inzhenerno-FizicheskiiZhurnal, Vol. 81, No. 6, pp. 1034–1045, November–December, 2008. Original article submitted February 28, 2007; revision submitted April 15, 2008.

 1062-0125/08/8106-1076©2008 Springer Science+Business Media, Inc.

$$\varepsilon^2 C^{(m)}\partial_t T^{(m)} = \varepsilon^2 L_2^{(m)}(T^{(m)}) + \varepsilon L_1^{(m)}(T^{(m)}) + \partial_3\left(\lambda_{33}^{(m)}\partial_3 T^{(m)}\right) + \varepsilon^2 Q^{(m)}(\mathbf{x}, t)\,, \quad \mathbf{x} = \{x_1, x_2, x_3\}\,, \tag{2}$$

where

$$L_1^{(m)}(T^{(m)}) \equiv \partial_1\left(\lambda_{13}^{(m)}\partial_3 T^{(m)}\right) + \partial_2\left(\lambda_{23}^{(m)}\partial_3 T^{(m)}\right) + \partial_3\left(\lambda_{31}^{(m)}\partial_1 T^{(m)} + \lambda_{32}^{(m)}\partial_2 T^{(m)}\right);$$

$$L_2^{(m)}(T^{(m)}) \equiv \partial_1\left(\lambda_{11}^{(m)}\partial_1 T^{(m)} + \lambda_{12}^{(m)}\partial_2 T^{(m)}\right) + \partial_2\left(\lambda_{21}^{(m)}\partial_1 T^{(m)} + \lambda_{22}^{(m)}\partial_2 T^{(m)}\right); \tag{3}$$

the conditions of conjugation of the solution for the heat flux and temperature on the surfaces $x_3 = H_m$ of contact of the mth and $(m+1)$th layers

$$\varepsilon\left(\lambda_{31}^{(m)}\partial_1 T^{(m)} + \lambda_{32}^{(m)}\partial_2 T^{(m)}\right) + \lambda_{33}^{(m)}\partial_3 T^{(m)} = \varepsilon\left(\lambda_{31}^{(n)}\partial_1 T^{(n)} + \lambda_{32}^{(n)}\partial_2 T^{(n)}\right)$$

$$+ \lambda_{33}^{(n)}\partial_3 T^{(n)}\,, \quad T^{(m)} = T^{(n)}\,, \quad x_3 = H_m\,, \quad (x_1, x_2) \in G\,, \quad n = m+1\,, \quad 1 \le m \le M-1\,; \tag{4}$$

the boundary conditions of the general form that are specified on the faces of the plate:

$$\beta^{(-)}\left[\varepsilon\left(\lambda_{31}^{(1)}\partial_1 T^{(1)} + \lambda_{32}^{(1)}\partial_2 T^{(1)}\right) + \lambda_{33}^{(1)}\partial_3 T^{(1)}\right] = \varepsilon\gamma^{(-)}Q^{(-)} + \varepsilon\delta^{(-)}\alpha_{(-)}\left(T^{(1)} - T_\infty^{(-)}\right), \quad x_3 = 0\,, \quad (x_1, x_2) \in G\,,$$

$$-\beta^{(+)}\left[\varepsilon\left(\lambda_{31}^{(M)}\partial_1 T^{(M)(1)} + \lambda_{32}^{(M)}\partial_2 T^{(M)}\right) + \lambda_{33}^{(M)}\partial_3 T^{(M)}\right] = \varepsilon\gamma^{(+)}Q^{(+)} + \varepsilon\delta^{(+)}\alpha_{(+)}\left(T^{(M)} - T_\infty^{(+)}\right), \tag{5}$$

$$x_3 = H_M\,, \quad (x_1, x_2) \in G\,;$$

the boundary conditions specified on the end surface (edge) of the plate:

$$-\beta\varepsilon\sum_{p=1}^{2} n_p \sum_{l=1}^{2} \lambda_{pl}^{(m)}\partial_l T^{(m)} - \beta\left(n_1\lambda_{13}^{(m)} + n_2\lambda_{23}^{(m)}\right)\partial_3 T^{(m)} = \varepsilon\gamma q_n^{(m)}$$

$$+ \varepsilon\delta\alpha^{(m)}\left(T^{(m)} - T_\infty\right), \quad (x_1, x_2) \in \Gamma\,, \quad H_{m-1} \le x_3 \le H_m\,, \quad t \ge t_0\,, \quad 1 \le m \le M\,, \tag{6}$$

and the initial condition at the instant of time $t = t_0$:

$$T^{(m)}(\mathbf{x}, t_0) = T_0^{(m)}(\mathbf{x})\,, \quad H_{m-1} \le x_3 \le H_m\,, \quad (x_1, x_2) \in G\,, \quad 1 \le m \le M\,. \tag{7}$$

If it is assumed that the variation in the small geometric parameter ε corresponds to the change in the plate thickness $\overline{H}$ (the quantities $\overline{H}_m$ vary in proportion to the change in $\overline{H}$, i.e., $H_m = \overline{H}_m/\overline{H} = \text{const}$) with a fixed geometry of the structure in plan (with a fixed characteristic dimension a), the basic functions and quantities given in (1) have the following asymptotic properties:

$$\lambda_{ij}^{(m)} = O(1) \quad (i, j = 1, 2, 3)\,, \quad Q^{(m)} = O(1)\,, \quad T_\infty^{(\pm)} = O(1)\,, \quad Q^{(+)} = O(1)\,, \quad T_\infty = O(1)\,,$$

$$q_n^{(m)} = O(1)\,, \quad \alpha^{(m)} = O(1)\,, \quad C^{(m)} = O(1)\,, \quad T_0^{(m)} = O(1) \quad \text{for} \quad \varepsilon \to 0\,. \tag{8}$$

In boundary conditions of the general form (5), the dimensionless coefficients $\alpha_{(\pm)}$ characterizing the Biot number [1] on the faces can be of the order of unity or can be larger or small quantities compared to unity [3, 4].

Asymptotic Analysis of the Problem of Heat Conduction of Laminar Plates. In what follows, we will consider the case where the Biot numbers $\alpha_{(+)}$ and $\alpha_{(-)}$ are small independent parameters on both faces (we should take

$\beta^{(\pm)} = 1$ in (5)). Since $\alpha_{(+)}$ and $\alpha_{(-)}$ appear only in boundary conditions (5) and are factors of the functions $T^{(1)}$ and $T^{(M)}$ rather than of their derivatives and the asymptotic properties analogous to (8) for $\alpha_{(+)} \to 0$ and $\alpha_{(-)} \to 0$ hold for the basic functions and quantities given in (1), the boundary-value problem (2)–(7) is a problem with a regular perturbation in parameters $\alpha_{(+)}$ and $\alpha_{(-)}$. To simplify the solution of this initial boundary-value problem we use the asymptotic expansion

$$T^{(m)}(x_1, x_2, x_3, t) \sim \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} T_{ij}^{(m)}(x_1, x_2, x_3, t)\, \alpha_{(-)}^{i} \alpha_{(+)}^{j}, \quad 0 \le \alpha_{(-)}, \ \ \alpha_{(+)} < 1 . \tag{9}$$

Substituting (9) into (2)–(7) and collecting terms with the same degrees $\alpha_{(-)}^{i}\alpha_{(+)}^{i}$, we obtain the following chain of equalities:

$$\varepsilon^2 C^{(m)} \partial_t T_{ij}^{(m)} = \varepsilon^2 L_2^{(m)}\left(T_{ij}^{(m)}\right) + \varepsilon L_1^{(m)}\left(T_{ij}^{(m)}\right) + \partial_3\left(\lambda_{33}^{(m)} \partial_3 T_{ij}^{(m)}\right) + \delta_{0i}\delta_{0j}\varepsilon^2 Q^{(m)}(\mathbf{x}, t)\,; \tag{10}$$

$$\varepsilon\left(\lambda_{31}^{(m)}\partial_1 T_{ij}^{(m)} + \lambda_{32}^{(m)}\partial_2 T_{ij}^{(m)}\right) + \lambda_{33}^{(m)}\partial_3 T_{ij}^{(m)} = \varepsilon\left(\lambda_{31}^{(n)}\partial_1 T_{ij}^{(n)} + \lambda_{32}^{(n)}\partial_2 T_{ij}^{(n)}\right)$$
$$+ \lambda_{33}^{(n)}\partial_3 T_{ij}^{(n)}, \quad T_{ij}^{(m)} = T_{ij}^{(n)}, \quad x_3 = H_m, \quad (x_1, x_2) \in G, \quad n = m+1, \quad 1 \le m \le M-1\,; \tag{11}$$

$$\varepsilon\left(\lambda_{31}^{(1)}\partial_1 T_{ij}^{(1)} + \lambda_{32}^{(1)}\partial_2 T_{ij}^{(1)}\right) + \lambda_{33}^{(1)}\partial_3 T_{ij}^{(1)} = \varepsilon\delta_{0i}\delta_{0j}\gamma^{(-)}Q^{(-)} - \varepsilon\delta^{(-)}\delta_{1i}\delta_{0j}T_\infty^{(-)}$$
$$+ \varepsilon\delta^{(-)}T_{i-1,j}^{(1)}, \quad x_3 = 0, \quad -\left[\varepsilon\left(\lambda_{31}^{(M)}\partial_1 T_{ij}^{(M)} + \lambda_{32}^{(M)}\partial_2 T_{ij}^{(M)}\right) + \lambda_{33}^{(M)}\partial_3 T_{ij}^{(M)}\right]$$
$$= \varepsilon\delta_{0i}\delta_{0j}\gamma^{(+)}Q^{(+)} - \varepsilon\delta_{0i}\delta_{1j}\delta^{(+)}T_\infty^{(+)} + \varepsilon\delta^{(+)}T_{i,j-1}^{(M)}, \quad x_3 = H_M, \quad (x_1, x_2) \in G\,; \tag{12}$$

$$-\beta\varepsilon\sum_{p=1}^{2} n_p \sum_{l=1}^{2} \lambda_{pl}^{(m)}\partial_l T_{ij}^{(m)} - \beta\left(n_1\lambda_{13}^{(m)} + n_2\lambda_{23}^{(m)}\right)\partial_3 T_{ij}^{(m)} = \varepsilon\delta\alpha^{(m)}T_{ij}^{(m)}$$
$$+ \varepsilon\delta_{0i}\delta_{0j}\left(\gamma q_n^{(m)} - \delta\alpha^{(m)}T_\infty\right), \quad (x_1, x_2) \in \Gamma, \quad H_{m-1} \le x_3 \le H_m, \quad t \ge t_0, \quad 1 \le m \le M\,; \tag{13}$$

$$T_{ij}^{(m)}(\mathbf{x}, t_0) = \delta_{0i}\delta_{0j}T_0^{(m)}(\mathbf{x}), \quad H_{m-1} \le x_3 \le H_m, \quad (x_1, x_2) \in G, \quad 1 \le m \le M, \quad i, j \ge 0\,, \tag{14}$$

where δ_{kl} is the Kronecker symbol. In the first equality of (12) for $i = 0$ and in the second equality of (12) for $j = 0$, we should allow for

$$T_{-1,j}^{(m)} \equiv 0, \quad T_{i,-1}^{(m)} \equiv 0, \quad 1 \le m \le M, \quad i, j = 0, 1, 2, 3, \ldots . \tag{15}$$

The initial boundary-value problems (10)–(14) with account for (15) can successively be integrated for all i and $j \ge 0$, i.e., we can successively determine all the coefficients of expansion (9).

The presence of the small geometric parameter ε of the higher derivatives in Eq. (10), in conjugation conditions (11), and in boundary conditions (12) and (13) points to the fact that the initial boundary-value problem (10)–(14), for any i and $j \ge 0$, is a problem with singular perturbation; therefore, the solution of this problem should be sought in the form

$$T_{ij}^{(m)} = T_{*ij}^{(m)} + T_{\tau ij}^{(m)} + T_{bij}^{(m)}, \quad 1 \le m \le M, \quad i, j = 0, 1, 2, 3, \ldots, \tag{16}$$

where $T_{*ij}^{(m)}$ is the external asymptotic expansion of the function $T_{ij}^{(m)}$, which characterizes the basic temperature field in the mth layer, $T_{\tau ij}^{(m)}$ is the correction to the external expansion in the vicinity of the initial instant of time $t = t_0$, and $T_{bij}^{(m)}$ is the correction to the external expansion in the boundary layer in the vicinity of the plate's end surface.

In what follows, the present investigation seeks to determine the external asymptotic expansion $T_{*ij}^{(m)}$. To obtain a noncontradictory chain of equalities for determining $T_{*ij}^{(m)}$ we should prescribe the asymptotic expansion in the form

$$T_{*ij}^{(m)}(x_1, x_2, x_3, t) \sim \frac{1}{\varepsilon^{i+j+1}} \sum_{k=0}^{\infty} T_{ijk}^{(m)}(x_1, x_2, x_3, t)\, \varepsilon^k, \quad 1 \le m \le M, \quad i, j \ge 0. \tag{17}$$

Let us substitute (17) into (10)–(14) and collect terms with the same degrees of ε; then we obtain the following chain of equalities for determination of the functions $T_{ijk}^{(m)}(\mathbf{x}, t)$:

$$\partial_3\left(\lambda_{33}^{(m)}\partial_3 T_{ij0}^{(m)}\right) = 0, \quad 1 \le m \le M; \tag{18}$$

$$\lambda_{33}^{(m)}\partial_3 T_{ij0}^{(m)} = \lambda_{33}^{(n)}\partial_3 T_{ij0}^{(n)}, \quad T_{ij0}^{(m)}(\mathbf{x}, t) = T_{ij0}^{(n)}(\mathbf{x}, t), \quad x_3 = H_m, \quad n = m+1, \quad 1 \le m \le M-1; \tag{19}$$

$$\lambda_{33}^{(1)}\partial_3 T_{ij0}^{(1)} = 0, \quad x_3 = 0, \quad -\lambda_{33}^{(M)}\partial_3 T_{ij0}^{(M)} = 0, \quad x_3 = H_M, \quad (x_1, x_2) \in G; \tag{20}$$

$$-\beta\left(n_1\lambda_{13}^{(m)} + n_2\lambda_{23}^{(m)}\right)\partial_3 T_{ij0}^{(m)} = 0, \quad (x_1, x_2) \in \Gamma, \quad H_{m-1} \le x_3 \le H_m, \quad 1 \le m \le M; \tag{21}$$

$$T_{ij0}^{(m)}(\mathbf{x}, t_0) = 0, \quad H_{m-1} \le x_3 \le H_m, \quad (x_1, x_2) \in G, \quad 1 \le m \le M; \tag{22}$$

$$\partial_3\left(\lambda_{33}^{(m)}\partial_3 T_{ij1}^{(m)}\right) + L_1^{(m)}\left(T_{ij0}^{(m)}\right) = 0, \quad 1 \le m \le M; \tag{23}$$

$$\begin{gathered}\lambda_{33}^{(m)}\partial_3 T_{ij1}^{(m)} + \lambda_{31}^{(m)}\partial_1 T_{ij0}^{(m)} + \lambda_{32}^{(m)}\partial_2 T_{ij0}^{(m)} = \lambda_{33}^{(n)}\partial_3 T_{ij1}^{(n)} + \lambda_{31}^{(n)}\partial_1 T_{ij0}^{(n)} + \lambda_{32}^{(n)}\partial_2 T_{ij0}^{(n)}, \\ T_{ij1}^{(m)}(\mathbf{x}, t) = T_{ij1}^{(n)}(\mathbf{x}, t), \quad x_3 = H_m, \quad (x_1, x_2) \in G, \quad n = m+1, \quad 1 \le m \le M-1;\end{gathered} \tag{24}$$

$$\begin{gathered}\lambda_{33}^{(1)}\partial_3 T_{ij1}^{(1)} + \lambda_{31}^{(1)}\partial_1 T_{ij0}^{(1)} + \lambda_{32}^{(1)}\partial_2 T_{ij0}^{(1)} = 0, \quad x_3 = 0, \\ -\left(\lambda_{33}^{(M)}\partial_3 T_{ij1}^{(M)} + \lambda_{31}^{(M)}\partial_1 T_{ij0}^{(M)} + \lambda_{32}^{(M)}\partial_2 T_{ij0}^{(M)}\right) = 0, \quad x_3 = H_M, \quad (x_1, x_2) \in G;\end{gathered} \tag{25}$$

$$\begin{gathered}-\beta\sum_{p=1}^{2} n_p \sum_{l=1}^{2} \lambda_{pl}^{(m)}\partial_l T_{ij0}^{(m)} - \beta\left(n_1\lambda_{13}^{(m)} + n_2\lambda_{23}^{(m)}\right)\partial_3 T_{ij1}^{(m)} - \delta\alpha^{(m)} T_{ij0}^{(m)} = 0, \\ (x_1, x_2) \in \Gamma, \quad H_{m-1} \le x_3 \le H_m, \quad 1 \le m \le M;\end{gathered} \tag{26}$$

$$T_{ij1}^{(m)}(\mathbf{x}, t_0) = \delta_{0i}\delta_{0j} T_0^{(m)}(\mathbf{x}), \quad H_{m-1} \le x_3 \le H_m, \quad (x_1, x_2) \in G, \quad 1 \le m \le M; \tag{27}$$

$$\partial_3\left(\lambda_{33}^{(m)}\partial_3 T_{ij2}^{(m)}\right) + L_1^{(m)}\left(T_{ij1}^{(m)}\right) + L_2^{(m)}\left(T_{ij0}^{(m)}\right) - C^{(m)}\partial_t T_{ij0}^{(m)} = 0, \quad 1 \le m \le M; \tag{28}$$

$$\lambda_{33}^{(m)}\partial_3 T_{ij2}^{(m)} + \lambda_{31}^{(m)}\partial_1 T_{ij1}^{(m)} + \lambda_{32}^{(m)}\partial_2 T_{ij1}^{(m)} = \lambda_{33}^{(n)}\partial_3 T_{ij2}^{(n)} + \lambda_{31}^{(n)}\partial_1 T_{ij1}^{(n)} + \lambda_{32}^{(n)}\partial_2 T_{ij1}^{(n)} ,$$

$$T_{ij2}^{(m)}(\mathbf{x}, t) = T_{ij2}^{(n)}(\mathbf{x}, t) , \quad x_3 = H_m , \quad (x_1, x_2) \in G , \quad n = m + 1 , \quad 1 \le m \le M - 1 ; \tag{29}$$

$$\lambda_{33}^{(1)}\partial_3 T_{ij2}^{(1)} + \lambda_{31}^{(1)}\partial_1 T_{ij1}^{(1)} + \lambda_{32}^{(1)}\partial_2 T_{ij1}^{(1)} = \delta_{0i}\delta_{0j}\gamma^{(-)}Q^{(-)} + \delta^{(-)}T_{i-1,j,0}^{(1)} , \quad x_3 = 0 ,$$

$$-\left(\lambda_{33}^{(M)}\partial_3 T_{ij2}^{(M)} + \lambda_{31}^{(M)}\partial_1 T_{ij1}^{(M)} + \lambda_{32}^{(M)}\partial_2 T_{ij1}^{(M)}\right) = \delta_{0i}\delta_{0j}\gamma^{(+)}Q^{(+)} + \delta^{(+)}T_{i,j-1,0}^{(M)} , \quad x_3 = H_M , \quad (x_1, x_2) \in G ; \tag{30}$$

$$-\beta \sum_{p=1}^{2} n_p \sum_{l=1}^{2} \lambda_{pl}^{(m)}\partial_l T_{ij1}^{(m)} - \beta\left(n_1\lambda_{13}^{(m)} + n_2\lambda_{23}^{(m)}\right)\partial_3 T_{ij2}^{(m)} - \delta\alpha^{(m)}T_{ij1}^{(m)}$$

$$= \delta_{0i}\delta_{0j}\left(\gamma q_n^{(m)} - \delta\alpha^{(m)}T_\infty\right), \quad (x_1, x_2) \in \Gamma , \quad H_{m-1} \le x_3 \le H_m , \quad 1 \le m \le M ; \tag{31}$$

$$T_{ij2}^{(m)}(\mathbf{x}, t_0) = 0 , \quad H_{m-1} \le x_3 \le H_m , \quad (x_1, x_2) \in G , \quad 1 \le m \le M ; \tag{32}$$

$$\partial_3\left(\lambda_{33}^{(m)}\partial_3 T_{ijk}^{(m)}\right) + L_1^{(m)}\left(T_{ijk-1}^{(m)}\right) + L_2^{(m)}\left(T_{ijk-2}^{(m)}\right) - C^{(m)}\partial_t T_{ijk-2}^{(m)} = -\,\delta_{0i}\delta_{0j}\delta_{3k}Q^{(m)}(\mathbf{x}, t) ; \tag{33}$$

$$\lambda_{33}^{(m)}\partial_3 T_{ijk}^{(m)} + \lambda_{31}^{(m)}\partial_1 T_{ijk-1}^{(m)} + \lambda_{32}^{(m)}\partial_2 T_{ijk-1}^{(m)} = \lambda_{33}^{(n)}\partial_3 T_{ijk}^{(n)} + \lambda_{31}^{(n)}\partial_1 T_{ijk-1}^{(n)} + \lambda_{32}^{(n)}\partial_2 T_{ijk-1}^{(n)} ,$$

$$T_{ijk}^{(m)}(\mathbf{x}, t) = T_{ijk}^{(n)}(\mathbf{x}, t) , \quad x_3 = H_m , \quad (x_1, x_2) \in G , \quad n = m + 1 , \quad 1 \le m \le M - 1 ; \tag{34}$$

$$\lambda_{33}^{(1)}\partial_3 T_{ijk}^{(1)} + \lambda_{31}^{(1)}\partial_1 T_{ijk-1}^{(1)} + \lambda_{32}^{(1)}\partial_2 T_{ijk-1}^{(1)} = \delta^{(-)}T_{i-1,j,k-2}^{(1)} - \delta_{1i}\delta_{0j}\delta_{3k}\delta^{(-)}T_\infty^{(-)} , \quad x_3 = 0 ,$$

$$-\left(\lambda_{33}^{(M)}\partial_3 T_{ijk}^{(M)} + \lambda_{31}^{(M)}\partial_1 T_{ijk-1}^{(M)} + \lambda_{32}^{(M)}\partial_2 T_{ijk-1}^{(M)}\right) = \delta^{(+)}T_{i,j-1,k-2}^{(M)} - \delta_{0i}\delta_{1j}\delta_{3k}\delta^{(+)}T_\infty^{(+)} , \quad x_3 = H_M , \quad (x_1, x_2) \in G ; \tag{35}$$

$$-\beta \sum_{p=1}^{2} n_p \sum_{l=1}^{2} \lambda_{pl}^{(m)}\partial_l T_{ijk-1}^{(m)} - \beta\left(n_1\lambda_{13}^{(m)} + n_2\lambda_{23}^{(m)}\right)\partial_3 T_{ijk}^{(m)} - \delta\alpha^{(m)}T_{ijk-1}^{(m)} = 0 ,$$

$$(x_1, x_2) \in \Gamma , \quad H_{m-1} \le x_3 \le H_m , \quad 1 \le m \le M , \quad k = 3, 4, 5, \ldots ; \tag{36}$$

$$T_{ijk}^{(m)}(\mathbf{x}, t_0) = 0 , \quad H_{m-1} \le x_3 \le H_m , \quad (x_1, x_2) \in G , \quad 1 \le m \le M , \quad k = 3, 4, 5, \ldots , \quad i, j = 0, 1, 2, \ldots , \tag{37}$$

where, according to (15) and (17), we should allow for

$$T_{-1,j,k}^{(m)} \equiv 0 , \quad T_{i,-1,k}^{(m)} \equiv 0 , \quad 1 \le m \le M , \quad i, j, k = 0, 1, 2, 3, \ldots . \tag{38}$$

Let us construct the solution of system (18)–(37). Integrating Eq. (18), with account for (19) and (20) and for $\lambda_{33}^{(m)} > 0$ (by virtue of the Onsager postulate [5]), we have

$$T_{ij0}^{(m)}(\mathbf{x}, t) = \theta_{ij0}(x_1, x_2, t) , \quad 1 \le m \le M , \tag{39}$$

where θ_{ij0} is the arbitrary function to be subsequently determined. Expression (39) yields the observance of boundary condition (21) on the plate's edge, whereas (22), with account for (39), yields the initial condition

$$\theta_{ij0}\,(x_1, x_2, t_0) = 0, \quad 1 \le m \le M\,. \tag{40}$$

Integrating Eq. (23) with respect to the variable x_3, with account for (39), (3), (24), and (25), we obtain

$$\lambda_{33}^{(m)}\partial_3 T_{ij1}^{(m)} + \lambda_{31}^{(m)}\partial_1 T_{ij0}^{(m)} + \lambda_{32}^{(m)}\partial_2 T_{ij0}^{(m)} = 0\,, \tag{41}$$

whence we have

$$T_{ij1}^{(m)}\,(\mathbf{x}, t) = \theta_{ij1}\,(x_1, x_2, t) - F_{ij1}^{(m)}\,(\mathbf{x}, t)\,, \quad 1 \le m \le M\,, \tag{42}$$

where

$$F_{ij1}^{(m)}\,(\mathbf{x}, t) \equiv \int\limits_{H_{m-1}}^{x_3} \frac{\lambda_{31}^{(m)}\partial_1\theta_{ij0} + \lambda_{32}^{(m)}\partial_2\theta_{ij0}}{\lambda_{33}^{(m)}}\,dx_3 + \sum_{l=1}^{m-1}\int\limits_{H_{l-1}}^{H_l} \frac{\lambda_{31}^{(l)}\partial_1\theta_{ij0} + \lambda_{32}^{(l)}\partial_2\theta_{ij0}}{\lambda_{33}^{(l)}}\,dx_3\,; \tag{43}$$

$\theta_{ij1}(x_1,\ x_2,\ t) \equiv T_{ij1}^{(1)}(x_1,\ x_2,\ 0,\ t)$ is the arbitrary function to be determined. We express the derivative $\partial_3 T_{ij1}^{(m)}$ from (41) and substitute it into (26); then we obtain

$$-\,\beta\sum_{p=1}^{2} n_p \sum_{l=1}^{2}\lambda_{pl}^{(m)}\partial_l\theta_{ij0} + \beta\left(n_1\lambda_{13}^{(m)} + n_2\lambda_{23}^{(m)}\right)\frac{\lambda_{31}^{(m)}\partial_1\theta_{ij0} + \lambda_{32}^{(m)}\partial_2\theta_{ij0}}{\lambda_{33}^{(m)}}$$

$$-\,\delta\alpha^{(m)}\theta_{ij0} = 0\,, \quad (x_1, x_2) \in \Gamma\,, \quad H_{m-1} \le x_3 \le H_m\,, \quad 1 \le m \le M\,. \tag{44}$$

Since the materials of the layers are assumed to be arbitrary (and generally irregular in thickness) and the function θ_{ij0} is independent of the variable x_3, boundary condition (44) cannot be observed accurately at all points of the end surface of the plate; therefore, here and in what follows boundary conditions (44), (31), and (36) on the plate's edges will be observed in an integral sense (by integrating the above equalities over the plate's thickness), which is a necessary and sufficient condition for attenuation of boundary layers [1].

Integrating relation (44) over the plate's thickness, we obtain the boundary condition on the edge for the function θ_{ij0}:

$$\beta\sum_{l=1}^{2}\partial_l\theta_{ij0}\sum_{m=1}^{M}\int\limits_{H_{m-1}}^{H_m}\left[-\,n_1\lambda_{1l}^{(m)} - n_2\lambda_{2l}^{(m)} + \lambda_{3l}^{(m)}\left(n_1\lambda_{13}^{(m)} + n_2\lambda_{23}^{(m)}\right)\Big/\lambda_{33}^{(m)}\right]dx_3$$

$$-\,\delta\theta_{ij0}\sum_{m=1}^{M}\int\limits_{H_{m-1}}^{H_m}\alpha^{(m)}dx_3 = 0\,, \quad (x_1, x_2) \in \Gamma\,. \tag{45}$$

We substitute (42) into initial condition (27); then we obtain

$$\theta_{ij1}\,(x_1, x_2, t_0) - F_{ij1}^{(m)}\,(\mathbf{x}, t_0) = \delta_{0i}\delta_{0j}T_0^{(m)}\,(\mathbf{x})\,. \tag{46}$$

Since the initial temperature distribution $T_0^{(m)}(\mathbf{x})$ is arbitrary, the function θ_{ij1} is independent of the variable x_3 and the function $F_{ij1}^{(m)}$ of the variable x_3 has quite a definite dependence (43), initial condition (46) cannot generally be observed accurately at all points of the plate; therefore, here and in what follows initial conditions (46), (32), and (37) will be observed in an integral sense (by integrating these equalities over the plate's thickness), which is a necessary and sufficient condition for attenuation of the correction $T_{\tau ij}^{(m)}$ in (16) in the vicinity of the initial instant of time t_0.

Integrating equality (46) over the plate's thickness, we obtain, with allowance for $H_M = 1$ (see (1)), the initial condition for the function θ_{ij1}:

$$\theta_{ij1}(x_1, x_2, t_0) = \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \left[\delta_{0i}\delta_{0j} T_0^{(m)}(\mathbf{x}) + F_{ij1}^{(m)}(\mathbf{x}, t_0) \right] dx_3 , \quad (x_1, x_2) \in G . \tag{47}$$

Equation (28) with account for (3), (39), and (41) can be transformed to the form

$$\partial_3 \left(\lambda_{33}^{(m)} \partial_3 T_{ij2}^{(m)} + \lambda_{31}^{(m)} \partial_1 T_{ij1}^{(m)} + \lambda_{32}^{(m)} \partial_2 T_1^{(m)} \right) = - \sum_{l=1}^{2} \partial_l \left(\lambda_{l1}^{(m)} \partial_1 \theta_{ij0} + \lambda_{l2}^{(m)} \partial_2 \theta_{ij0} \right)$$

$$+ \sum_{l=1}^{2} \partial_l \left[\lambda_{l3}^{(m)} \left(\lambda_{31}^{(m)} \partial_1 \theta_{ij0} + \lambda_{32}^{(m)} \partial_2 \theta_{ij0} \right) \Big/ \lambda_{33}^{(m)} \right] + C^{(m)} \partial_t \theta_{ij0} .$$

Integrating this equation with respect to x_3, with account for (29) and (39) and for the first equality of (30), we obtain

$$\lambda_{33}^{(m)} \partial_3 T_{ij2}^{(m)} + \lambda_{31}^{(m)} \partial_1 T_{ij1}^{(m)} + \lambda_{32}^{(m)} \partial_2 T_{ij1}^{(m)} = Q_{ij2}^{(m)}(\mathbf{x}, t) , \quad 1 \le m \le M , \tag{48}$$

where

$$\begin{aligned} Q_{ij2}^{(m)}(\mathbf{x}, t) \equiv \delta_{0i}\delta_{0j}\gamma^{(-)}Q^{(-)} + \delta^{(-)}\theta_{i-1,j,0} + \int_{H_{m-1}}^{x_3} \Bigg\{ & \sum_{l=1}^{2} \partial_l \left[\lambda_{l3}^{(m)} \left(\lambda_{31}^{(m)} \partial_1 \theta_{ij0} + \lambda_{32}^{(m)} \partial_2 \theta_{ij0} \right) \Big/ \lambda_{33}^{(m)} \right] \\ & - \sum_{l=1}^{2} \partial_l \left(\lambda_{l1}^{(m)} \partial_1 \theta_{ij0} + \lambda_{l2}^{(m)} \partial_2 \theta_{ij0} \right) + C^{(m)} \partial_t \theta_{ij0} \Bigg\} dx_3 + D^{(m)}(\theta_{ij0}) ; \end{aligned} \tag{49}$$

the differential operator $D^{(m)}(\bullet)$ has the form

$$\begin{aligned} D^{(m)}(\bullet) \equiv \sum_{l=1}^{m-1} \int_{H_{l-1}}^{H_l} \Bigg\{ & \sum_{p=1}^{2} \partial_p \left[\lambda_{p3}^{(l)} \left(\lambda_{31}^{(l)} \partial_1 (\bullet) + \lambda_{32}^{(l)} \partial_2 (\bullet) \right) \Big/ \lambda_{33}^{(l)} \right] \\ & - \sum_{p=1}^{2} \partial_p \left(\lambda_{p1}^{(l)} \partial_1 (\bullet) + \lambda_{p2}^{(l)} \partial_2 (\bullet) \right) + C^{(m)} \partial_t (\bullet) \Bigg\} dx_3 , \quad D^{(1)}(\bullet) \equiv 0 . \end{aligned} \tag{50}$$

Equality (48) for $m = M$ and $x_3 = H_M = 1$ and the second equality of (30), with account for (49), (50), and (39), yield

$$D^{(M+1)}(\theta_{ij0}) = - \delta_{0i}\delta_{0j} \left(\gamma^{(-)}Q^{(-)} + \gamma^{(+)}Q^{(+)} \right) - \delta^{(-)}\theta_{i-1,j,0} - \delta^{(+)}\theta_{i,j-1,0} , \quad (x_1, x_2) \in G . \tag{51}$$

This equation, with account for (38) and (39), determines the function $\theta_{ij0}(x_1, x_2, t)$ for the already known $\theta_{i-1,j,0}$ and $\theta_{i,j-1,0}$. Equation (51) corresponds to boundary condition (45) specified on the plate's edge and to initial condition (40). Knowing the function θ_{ij0} from the initial boundary-value problem (40), (45), and (51), we obtain, by virtue of

(43), (49), and (50), the known right-hand side in (48) and function $F_{ij1}^{(m)}(\mathbf{x}, t)$ in (42). Substituting (42) into (48), we obtain

$$\partial_3 T_{ij2}^{(m)} = \left(Q_{ij2}^{(m)}(\mathbf{x}, t) + \lambda_{31}^{(m)}\partial_1 F_{ij1}^{(m)} + \lambda_{32}^{(m)}\partial_2 F_{ij1}^{(m)}\right) / \lambda_{33}^{(m)} - \left(\lambda_{31}^{(m)}\partial_1\theta_{ij1} + \lambda_{32}^{(m)}\partial_2\theta_{ij1}\right) / \lambda_{33}^{(m)}, \tag{52}$$

where the first term on the right-hand side is the known function. Integrating this equality with respect to x_3, with account for (29), we will have

$$T_{ij2}^{(m)}(\mathbf{x}, t) = \theta_{ij2}(x_1, x_2, t) - F_{ij2}^{(m)}(\mathbf{x}, t), \quad 1 \le m \le M, \tag{53}$$

where

$$F_{ij2}^{(m)}(\mathbf{x}, t) \equiv \int_{H_{m-1}}^{x_3} \frac{\lambda_{31}^{(m)}\partial_1\theta_{ij1} + \lambda_{32}^{(m)}\partial_2\theta_{ij1}}{\lambda_{33}^{(m)}} dx_3 + \sum_{l=1}^{m-1} \int_{H_{l-1}}^{H_l} \frac{\lambda_{31}^{(l)}\partial_1\theta_{ij1} + \lambda_{32}^{(l)}\partial_2\theta_{ij1}}{\lambda_{33}^{(l)}} dx_3$$
$$- \int_{H_{m-1}}^{x_3} \frac{Q_{ij2}^{(m)} + \lambda_{31}^{(m)}\partial_1 F_{ij1}^{(m)} + \lambda_{32}^{(m)}\partial_2 F_{ij1}^{(m)}}{\lambda_{33}^{(m)}} dx_3 - \sum_{l=1}^{m-1} \int_{H_{l-1}}^{H_l} \frac{Q_{ij2}^{(l)} + \lambda_{31}^{(l)}\partial_1 F_{ij1}^{(l)} + \lambda_{32}^{(l)}\partial_2 F_{ij1}^{(l)}}{\lambda_{33}^{(l)}} dx_3; \tag{54}$$

$\theta_{ij2}(x_1, x_2, t) \equiv T_{ij2}^{(1)}(x_1, x_2, 0, t)$ is the arbitrary function to be determined.

We substitute (52) into boundary condition (31) on the edge and integrate it over the plate's thickness with account for (42); then we have

$$\beta \sum_{l=1}^{2} \partial_l \theta_{ij1} \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \left[-n_1\lambda_{1l}^{(m)} - n_2\lambda_{2l}^{(m)} + \lambda_{3l}^{(m)}\left(n_1\lambda_{13}^{(m)} + n_2\lambda_{23}^{(m)}\right) / \lambda_{33}^{(m)} \right] dx_3$$
$$- \delta\theta_{ij1} \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \alpha^{(m)} dx_3 = \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \left[-\beta \sum_{l=1}^{2} n_l \left(\lambda_{1l}^{(m)}\partial_1 F_{ij1}^{(m)} + \lambda_{l2}^{(m)}\partial_2 F_{ij1}^{(m)}\right) \right.$$
$$\left. + \beta\left(n_1\lambda_{13}^{(m)} + n_2\lambda_{23}^{(m)}\right)\left(Q_{ij2}^{(m)} + \lambda_{31}^{(m)}\partial_1 F_{ij1}^{(m)} + \lambda_{32}^{(m)}\partial_2 F_{ij1}^{(m)}\right) / \lambda_{33}^{(m)} + \delta_{0i}\delta_{0j}\left(\gamma q_n^{(m)} - \delta\alpha^{(m)} T_\infty\right) - \delta\alpha^{(m)} F_{ij1}^{(m)} \right] dx_3. \tag{55}$$

We substitute (53) into initial condition (32) and integrate it over the plate's thickness; then, with allowance for $H_M = 1$, we obtain

$$\theta_{ij2}(x_1, x_2, t_0) = \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} F_{ij2}^{(m)}(\mathbf{x}, t_0)\, dx_3, \quad (x_1, x_2) \in G. \tag{56}$$

By virtue of the formal similarity of relations (28)–(32) and (33)–(37) respectively and equalities (53), (42), and (48), (41), we can make, for the initial boundary-value problem (33)–(37) with $k \ge 3$, the assumptions

$$\lambda_{33}^{(m)}\partial_3 T_{ijk-1}^{(m)} + \lambda_{31}^{(m)}\partial_1 T_{ijk-2}^{(m)} + \lambda_{32}^{(m)}\partial_2 T_{ijk-2}^{(m)} = Q_{ijk-1}^{(m)}(\mathbf{x}, t), \quad 1 \le m \le M; \tag{57}$$

$$T_{ijk-2}^{(m)}(\mathbf{x}, t) = \theta_{ijk-2}(x_1, x_2, t) - F_{ijk-2}^{(m)}(\mathbf{x}, t), \quad 1 \le m \le M, \tag{58}$$

where $Q^{(m)}_{ijk-1}(\mathbf{x}, t)$ and $F^{(m)}_{ijk-2}(\mathbf{x}, t)$ are assumed to be already known functions. When $k = 3$ the assumptions (57) and (58) are fulfilled, since equalities (42) and (48) hold true and the functions $Q^{(m)}_{ij2}$ and $F^{(m)}_{ij1}$ are known from (43) and (49) and from the solved initial boundary-value problem (40), (45), and (51).

Let us express the derivative $\partial_3 T^{(m)}_{ijk-1}$ from (57) and substitute it into Eq. (33) with account for (3); then, after using equality (58), we obtain

$$\partial_3\left(\lambda^{(m)}_{33}\partial_3 T^{(m)}_{ijk} + \lambda^{(m)}_{31}\partial_1 T^{(m)}_{ijk-1} + \lambda^{(m)}_{32}\partial_2 T^{(m)}_{ijk-1}\right) = -\,\delta_{0i}\delta_{0j}\delta_{3k}Q^{(m)}(\mathbf{x}, t)$$

$$-\sum_{l=1}^{2}\left\{\partial_l\left[\lambda^{(m)}_{l3}\left(Q^{(m)}_{ijk-1} + \lambda^{(m)}_{31}\partial_1 F^{(m)}_{ijk-2} + \lambda^{(m)}_{32}\partial_2 F^{(m)}_{ijk-2}\right)\Big/\lambda^{(m)}_{33}\right] - \partial_l\left(\lambda^{(m)}_{1l}\partial_1 F^{(m)}_{ijk-2} + \lambda^{(m)}_{l2}\partial_2 F^{(m)}_{ijk-2}\right)\right\} - C^{(m)}\partial_t F^{(m)}_{ijk-2}$$

$$+\sum_{l=1}^{2}\left\{\partial_l\left[\lambda^{(m)}_{l3}\left(\lambda^{(m)}_{31}\partial_1\theta_{ijk-2} + \lambda^{(m)}_{32}\partial_2\theta_{ijk-2}\right)\Big/\lambda^{(m)}_{33}\right] - \partial_l\left(\lambda^{(m)}_{1l}\partial_1\theta_{ijk-2} + \lambda^{(m)}_{l2}\partial_2\theta_{ijk-2}\right)\right\} + C^{(m)}\partial_t\theta_{ijk-2}\,.$$

Integrating this equation with respect to x_3, with account for (34) and (35), we will have

$$\lambda^{(m)}_{33}\partial_3 T^{(m)}_{ijk} + \lambda^{(m)}_{31}\partial_1 T^{(m)}_{ijk-1} + \lambda^{(m)}_{32}\partial_2 T^{(m)}_{ijk-1} = Q^{(m)}_{ijk}(\mathbf{x}, t)\,, \quad 1 \le m \le M\,, \tag{59}$$

where

$$Q^{(m)}_{ijk}(\mathbf{x}, t) \equiv \delta^{(-)}T^{(1)}_{i-1,j,k-2}\big|_{x_3=0} - \delta_{1i}\delta_{0j}\delta_{3k}\delta^{(-)}T^{(-)}_{\infty} - \int_{H_{m-1}}^{x_3}\left\{\delta_{0i}\delta_{0j}\delta_{3k}Q^{(m)}(\mathbf{x}, t)\right.$$

$$-\sum_{p=1}^{2}\partial_p\left(\lambda^{(m)}_{1p}\partial_1 F^{(m)}_{ijk-2} + \lambda^{(m)}_{p2}\partial_2 F^{(m)}_{ijk-2}\right) + C^{(m)}\partial_t F^{(m)}_{ijk-2} + \sum_{p=1}^{2}\partial_p\left[\lambda^{(m)}_{p3}\left(Q^{(m)}_{ijk-1} + \lambda^{(m)}_{31}\right.\right.$$

$$\left.\left.\left.\times\,\partial_1 F^{(m)}_{ijk-2} + \lambda^{(m)}_{32}\partial_2 F^{(m)}_{ijk-2}\right)\Big/\lambda^{(m)}_{33}\right]\right\}dx_3 - \sum_{l=1}^{m-1}\int_{H_{l-1}}^{H_l}\left\{\delta_{0i}\delta_{0j}\delta_{3k}Q^{(l)}(\mathbf{x}, t) - \sum_{p=1}^{2}\partial_p\left(\lambda^{(l)}_{1p}\right.\right.$$

$$\left.\times\,\partial_1 F^{(l)}_{ijk-2} + \lambda^{(l)}_{p2}\partial_2 F^{(l)}_{ijk-2}\right) + C^{(l)}\partial_t F^{(l)}_{ijk-2} + \sum_{p=1}^{2}\partial_p\left[\lambda^{(l)}_{p3}\left(Q^{(l)}_{ijk-1} + \lambda^{(l)}_{31}\partial_1 F^{(l)}_{ijk-2}\right.\right.$$

$$\left.\left.\left.+\,\lambda^{(l)}_{32}\partial_2 F^{(l)}_{ijk-2}\right)\Big/\lambda^{(l)}_{33}\right]\right\}dx_3 + \int_{H_{m-1}}^{x_3}\left\{\sum_{p=1}^{2}\partial_p\left[\lambda^{(m)}_{p3}\left(\lambda^{(m)}_{31}\partial_1\theta_{ijk-2} + \lambda^{(m)}_{32}\partial_2\theta_{ijk-2}\right)\Big/\lambda^{(m)}_{33}\right]\right.$$

$$\left.-\sum_{p=1}^{2}\partial_p\left(\lambda^{(m)}_{1p}\partial_1\theta_{ijk-2} + \lambda^{(m)}_{p2}\partial_2\theta_{ijk-2}\right) + C^{(m)}\partial_t\theta_{ijk-2}\right\}dx_3 + D^{(m)}(\theta_{ijk-2})\,; \tag{60}$$

the differential operator $D^{(m)}(\bullet)$ has been determined in (50). Relation (59) for $m = M$, $x_3 = H_M = 1$, and the second equality of (35), with account for (50), yield

$$D^{(M+1)}(\theta_{ijk-2}) = -\,\delta^{(-)}T^{(1)}_{i-1,j,k-2}\big|_{x_3=0} - \delta^{(+)}T^{(M)}_{i,j-1,k-2}\big|_{x_3=1} + \delta_{1i}\delta_{0j}\delta_{3k}\delta^{(-)}T^{(-)}_{\infty}$$

$$+\,\delta_{0i}\delta_{1j}\delta_{3k}\delta^{(+)}T^{(+)}_{\infty} + \sum_{m=1}^{M}\int_{H_{m-1}}^{H_m}\left\{\delta_{0i}\delta_{0j}\delta_{3k}Q^{(m)}(\mathbf{x}, t) - \sum_{p=1}^{2}\partial_p\left(\lambda^{(m)}_{1p}\partial_1 F^{(m)}_{ijk-2} + \lambda^{(m)}_{p2}\partial_2 F^{(m)}_{ijk-2}\right)\right.$$

$$+\sum_{p=1}^{2}\partial_p\left[\lambda_{p3}^{(m)}\left(Q_{ijk-1}^{(m)}+\lambda_{31}^{(m)}\partial_1F_{ijk-2}^{(m)}+\lambda_{32}^{(m)}\partial_2F_{ijk-2}^{(m)}\right)/\lambda_{33}^{(m)}\right]+C^{(m)}\partial_tF_{ijk-2}^{(m)}\Bigg\}dx_3\,,\quad (x_1,x_2)\in G\,,\quad t\geq t_0\,, \tag{61}$$

where the right-hand side is the known function of the variables x_1, x_2, and t, since the function $Q_{ijk-1}^{(m)}$ and $F_{ijk-2}^{(m)}$ are assumed to be already known (see (57) and (58)).

When $k = 3$, Eq. (61) determines the function $\theta_{ij1}(x_1, x_2, t)$ with boundary condition (55) specified on the plate's edge and with initial condition (47). When $k \geq 4$ we obtain the boundary condition for Eq. (61)) by integrating equality (36) over the plate's thickness for the previous value of k (replacing k in (36) by $k-1$); then, eliminating the derivative $\partial_3 T_{ijk-1}^{(m)}$ from (57) and using equality (58), we will have

$$\beta\sum_{l=1}^{2}\partial_l\theta_{ijk-2}\sum_{m=1}^{M}\int_{H_{m-1}}^{H_m}\left[-n_1\lambda_{1l}^{(m)}-n_2\lambda_{2l}^{(m)}+\lambda_{3l}^{(m)}\left(n_1\lambda_{13}^{(m)}+n_2\lambda_{23}^{(m)}\right)/\lambda_{33}^{(m)}\right]dx_3-\delta\theta_{ijk-2}$$

$$\times\sum_{m=1}^{M}\int_{H_{m-1}}^{H_m}\alpha^{(m)}dx_3=\sum_{m=1}^{M}\int_{H_{m-1}}^{H_m}\Bigg[-\beta\sum_{l=1}^{2}n_l\left(\lambda_{1l}^{(m)}\partial_1F_{ijk-2}^{(m)}+\lambda_{l2}^{(m)}\partial_2F_{ijk-2}^{(m)}\right)+\beta\left(n_1\lambda_{13}^{(m)}+n_2\lambda_{23}^{(m)}\right)$$

$$\times\left(Q_{ijk-1}^{(m)}+\lambda_{31}^{(m)}\partial_1F_{ijk-2}^{(m)}+\lambda_{32}^{(m)}\partial_2F_{ijk-2}^{(m)}\right)/\lambda_{33}^{(m)}-\delta\alpha^{(m)}F_{ijk-2}^{(m)}\Bigg]dx_3\,,\quad k=4,5,6,\ldots\,. \tag{62}$$

To determine the initial condition corresponding to Eq. (61) for $k \geq 4$ we substitute (58) into equality (37) (replacing k by $k-2$ in it) and take into account that, according to the assumption made, the function $F_{ijk-2}^{(m)}$ is already known; then, after the integration over the plate's thickness, we obtain

$$\theta_{ijk-2}(x_1,x_2,t_0)=\sum_{m=1}^{M}\int_{H_{m-1}}^{H_m}F_{ijk-2}^{(m)}(\mathbf{x},t_0)\,dx_3\,,\quad (x_1,x_2)\in G\,,\quad k=4,5,6,\ldots\,. \tag{63}$$

We express the derivative $\partial_3 T_{ijk-1}^{(m)}$ from (57) and integrate the resulting equality with respect to x_3. As a result, with account for (58), we obtain

$$T_{ijk-1}^{(m)}(\mathbf{x},t)=\theta_{ijk-1}(x_1,x_2,t)-F_{ijk-1}^{(m)}(\mathbf{x},t)\,,\quad 1\leq m\leq M\,, \tag{64}$$

where

$$F_{ijk-1}^{(m)}(\mathbf{x},t)\equiv\int_{H_{m-1}}^{x_3}\frac{\lambda_{31}^{(m)}\partial_1\theta_{ijk-2}+\lambda_{32}^{(m)}\partial_2\theta_{ijk-2}}{\lambda_{33}^{(m)}}dx_3$$

$$+\sum_{l=1}^{m-1}\int_{H_{l-1}}^{H_l}\frac{\lambda_{31}^{(l)}\partial_1\theta_{ijk-2}+\lambda_{32}^{(l)}\partial_2\theta_{ijk-2}}{\lambda_{33}^{(l)}}dx_3-\int_{H_{m-1}}^{x_3}\frac{Q_{ijk-1}^{(m)}+\lambda_{31}^{(m)}\partial_1F_{ijk-2}^{(m)}+\lambda_{32}^{(m)}\partial_2F_{ijk-2}^{(m)}}{\lambda_{33}^{(m)}}dx_3$$

$$-\sum_{l=1}^{m-1}\int_{H_{l-1}}^{H_l}\frac{Q_{ijk-1}^{(l)}+\lambda_{31}^{(l)}\partial_1F_{ijk-2}^{(l)}+\lambda_{32}^{(l)}\partial_2F_{ijk-2}^{(l)}}{\lambda_{33}^{(l)}}dx_3\,; \tag{65}$$

$\theta_{ijk-1}(x_1, x_2, t) \equiv T_{ijk-1}^{(1)}(x_1, x_2, 0, t)$ is the arbitrary function to be determined. (When $k = 3$ equalities (64) and (65) agree with (53) and (54) respectively.)

Determining the function $\Theta^{(m)}_{ijk-2}$ from the initial boundary-value problem (61), (55), and (47) (for $k = 3$) or from (61)–(63) (for $k \geq 4$), we will have, by virtue of (65) and (60) and the assumptions (57) and (58), the known functions $F^{(m)}_{ijk-1}$ and $Q^{(m)}_{ijk}$ in equalities (59) and (64) which are formally in complete agreement with (57) and (58). Thus, the assumptions (57) and (58) also remain true for the next value of k; therefore; we can construct the solution of the initial boundary-value problem (33)–(37) (where i and $j = 0, 1, 2, ...$) for a new value of k, etc. according to the scheme (57)–(65).

The proposed algorithm of determination of the basic three-dimensional nonstationary temperature field in the laminar anisotropic plate shows that for the unknown coefficients $T^{(m)}_{ijk}$ in the asymptotic expansion (17) to be computed for each $k = 0, 1, 2, ...$ we must integrate the two-dimensional equations (51) and (61) which differ only in the known right-hand sides and in expanded form, by virtue of (50), appear as follows:

$$\sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} \left\{ \sum_{l=1}^{2} \partial_l \left[\lambda^{(m)}_{l3} \left(\lambda^{(m)}_{31} \partial_1 \theta_{ijk-2} + \lambda^{(m)}_{32} \partial_2 \theta_{ijk-2} \right) / \lambda^{(m)}_{33} \right] - \sum_{l=1}^{2} \partial_l \left(\lambda^{(m)}_{1l} \partial_1 \theta_{ijk-2} + \lambda^{(m)}_{l2} \partial_2 \theta_{ijk-2} \right) \right\} dx_3$$

$$= - \partial_t \theta_{ijk-2} \sum_{m=1}^{M} \int_{H_{m-1}}^{H_m} C^{(m)} dx_3 + W_k (x_1, x_2, t) \,, \qquad (66)$$

where W_k is determined by the right-hand side of (51) for $k = 2$ or by the right-hand side of (61) for $k \geq 3$.

Equation (66) contains the derivative with respect to the time t just of first order and the derivatives with respect to the spatial variables x_1 and x_2 of second order; consequently, it is a parabolic equation; the differential operator on its left-hand side is elliptic [3].

If the boundary conditions of the second kind ($\beta^{(\pm)} = \gamma^{(\pm)} = 1$ and $\delta^{(\pm)} = 0$, or $\alpha_{(\pm)} = 0$ in (5)) are specified on both faces, just the first term ($T^{(m)}(\mathbf{x}, t) = T^{(m)}_{00}(\mathbf{x}, t)$) is left in the expansion (9) and the external asymptotic expansion of the temperature will be determined by relation (17) for $i = j = 0$. The solution of the problem in question possesses the same asymptotic properties as those in [3] with the corresponding boundary conditions and renotation.

If we have convective heat exchange ($\beta^{(\pm)} = \delta^{(\pm)} = 1$ in (5)) with small Biot numbers on both faces and these numbers are of the order ε ($\alpha_{(\pm)} \sim \varepsilon$), by virtue of the expansions (9) and (17) we obtain the asymptotic properties of the basic temperature field $T^{(m)}_{*}$ that are analogous to those given in [3] for the cases of the boundary conditions of the second kind on the faces.

The external asymptotic expansion (constructed in the present work) of the temperature may lead to discrepancies in boundary conditions (6) on the plate's edges [6]; to eliminate these discrepancies we can use the regular procedure [1, 6] of introduction, in the vicinity of the contour Γ, of internal "extended" variables in the plane's plan which correspond to x_1 and x_2 and of construction of the internal asymptotic expansion for the boundary layer followed by its joining with the external expansion. Analogously we can eliminate the discrepancies in initial conditions (7), using the internal "extended" time variable in the vicinity of the initial instant t_0 and constructing an internal asymptotic expansion of the "boundary-layer" type in time followed by its joining with the external expansion. Study of these issues is beyond the scope of the present paper because of its limited volume.

Conclusions. The external asymptotic expansion obtained above can be used in strength and compliance analyses of thin-walled laminar structures, since the approximate theories of bending of plates used in practice (those of Kirchhof, Timoshenko, and others [7]) give an acceptable degree of accuracy only at a certain distance from the plate's edges or the lines of distortion of the stressed state, i.e., beyond the boundary layer and local effects propagating deep into the structure for a distance of the order of the plate's thickness [7]. Furthermore, the constructed asymptotic expansion of temperature can also be used in asymptotic analysis of the thermoelastic behavior of anisotropic plates. Thus, it has been assumed in [8] in studying the asymptotic properties of solutions of thermoelastic problems that the temperature in the plate beyond the boundary layer can be represented in the form of (17) for $i = j = 0$ but such representation has not been strictly substantiated.

This work was carried out with financial support from the Presidium of the Siberian Branch of the Russian Academy of Sciences (decree No. 54 of 09.02.06, project No. 2.2).

NOTATION

a, characteristic dimensions of the region occupied by the plate in the plane of real variables $\bar{x}_1$ and $\bar{x}_2$, m; $\bar{c}^{(m)}$, specific heat of the material of the mth layer, J/(kg·K); G, region occupied by the plate in the region of dimensionless variables x_1 and x_2 (in plan); $\bar{H}$, plate thickness, m; $\bar{H}_m$ = const > 0, z axis of the boundary between the mth and $(m+1)$th layers ($\bar{H} \equiv 0$ and $\bar{H}_M \equiv \bar{H}$), m; n_i, components of the vector of the unit normal to the end surface of the plate ($n_3 = 0$); $\bar{q}_n^{(m)}$, prescribed heat flux through the end surface of the mth layer, W/m^2; $\bar{Q}^{(\pm)}$, projections (prescribed on the faces ($\bar{x}_3 = 0$ and $\bar{H}$)) of the heat-flux vector onto the direction of the outer normal, W/m^2; $\bar{Q}^{(m)}$, power density of the internal heat sources in the mth layer, W/kg; $\bar{t}$, time, sec; $\bar{t}_0$, initial instant of time, sec; $\bar{t}_*$, characteristic time during which the process of nonstationary heat conduction is considered (it is selected so that the dimensionless heat capacity $C^{(m)}$ has the order of unity), sec; $\bar{T}^{(m)}$, temperature of the mth layer, K; $\bar{T}_\infty$, temperature of the ambient medium on the source of the end surface, K; $\bar{T}_\infty^{(\pm)}$, temperature of the ambient medium on the source side of the upper (+) and lower (-) faces of the plate, K; $\bar{T}_0^{(m)}$, initial temperature distribution in the mth layer, K; $\bar{T}_*$, certain characteristic value of the structural temperature (e.g., natural-state temperature), K; $\bar{\alpha}_{(\pm)}$, coefficients of convective heat exchange with the ambient medium on the upper (+) and lower (-) sides of the plate, W/(m^2·K); $\bar{\alpha}^{(m)}$, coefficient of heat exchange between the mth layer and the ambient medium on the end surface by the Newton law, W/(m^2·K); $\beta^{(\pm)}$, $\gamma^{(\pm)}$, and $\delta^{(\pm)}$, switching functions making it possible to specify one type of boundary conditions or another on the upper (+) and lower (-) faces; β, γ, and δ, switching functions making it possible to specify one type of boundary conditions or another on the end surface; Γ, contour bounding the region G; ε, small geometric parameter; $\bar{\lambda}_{ij}^{(m)}$, thermal conductivities of the material of the mth layer (generally functions of all spatial variables), W/(m·K); $\bar{\lambda}_*$, characteristic value of the thermal conductivity of the materials of the layers in the plate (e.g., maximum (over the layers) of the highest principal value of the thermal-conductivity tensor $\bar{\lambda}_{ij}^{(m)}$), W/(m·K); $\bar{\rho}^{(m)}$, bulk density of the material of the mth layer, kg/m^3; ∂_i, operator of partial differentiation with respect to the dimensionless spatial variable x_i (i = 1, 2, and 3); ∂_t, operator of partial differentiation with respect to the dimensionless time t. Superscripts: ($\pm$), values of the function on the upper (+) for $\bar{x}_3 = \bar{H}$ or lower (−) for $\bar{x}_3 = 0$ faces of the plate; *, characteristic value.

REFERENCES

1. I. E. Zino and É. A. Tropp, *Asymptotic Methods in Problems of Heat Conduction and Thermoelasticity Theory* [in Russian], Izd. Lenigradsk. Univ., Leningrad (1978).
2. Yu. V. Nemirovskii and A. P. Yankovskii, Refinement of asymptotic expansions of the problem of thermal conductivity of anisotropic plates, *Mat. Metody i Fiz.-Mat. Polya*, **48**, No. 2, 157–171 (2005).
3. Yu. V. Nemirovskii and A. P. Yankovskii, Method of asymptotic expansions of solutions for the problem of stationary heat conduction in lamellar anisotropic inhomogeneous plates, *Prikl. Mat. Mekh.*, **72**, Issue 1, 157–175 (2008).
4. A. V. Luikov, *Heat Conduction Theory* [in Russian], Vysshaya Shkola, Moscow (1967).
5. K. P. Gurov, *Phenomenological Thermodynamics of Irreversible Processes* [in Russian], Nauka, Moscow (1978).
6. A. M. Il'in, *Coordination of Asymptotic Expansions of Solutions of Boundary-Value Problems* [in Russian], Nauka, Moscow (1989).
7. S. A. Ambartsumyan, *The Theory of Anisotropic Plates* (*Strength*, *Stability*, *and Oscillations*) [in Russian], Nauka, Moscow (1967).
8. L. A. Agalovyan and R. S. Gevorkyan, *Nonclassical Boundary-Value Problems of Anisotropic Lamellar Beams*, *Plates*, *and Shells* [in Russian], Izd. Gitutyun NAN RA, Erevan (2005).